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THE HOMOTOPY TYPE OF THE SPACE OF DIFFEOMORPHISMS. II

BY

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ABSTRACT. The result (proved in Part I) that $\mathrm{Diff}(D^n, \partial) \simeq \Omega^{n+1}(\mathrm{PL}_n/O_n)$ is used to compute some new homotopy of $\mathrm{Diff}(D^n, \partial D^n)$. The relation between smooth and PL pseudo-isotopy is explored. Known and new results on the homotopy of PL_n are summarized.

5. Some remarks on PL_n and Top_n. We first summarize known results. In low dimensions we have the following are contractible:

(1) $\mathrm{PL}_1/O_1 \simeq \mathrm{PL}_2/O_2 \simeq *$, $\mathrm{Top}_1/O_1 \simeq \mathrm{Top}_2/O_2 \simeq *$.

It is trivial to show that $\mathrm{PL}_1/O_1 \simeq \mathrm{Top}_1/O_1 \simeq *$. Kneser [16] showed a long time ago that $\mathrm{Top}_2/O_2 \simeq *$. That $\mathrm{PL}_2/O_2 \simeq *$ has been shown recently by Akiba and P. Scott [1].

In the "stable" range, we recall the results of Haefliger and Wall [26a]:

(2) $\pi_i(\mathrm{PL}_{k+1}, \mathrm{PL}_k) = 0$ for $i \leq k-1$, and

(2′) $\mathrm{Ker}(\pi_{k-1}(\mathrm{PL}_k) \to \pi_{k-1}(\mathrm{PL}_{k+1})) = \mathrm{Image}\ i_*^k(\mathrm{Ker}(\pi_{k-1}(O_k) \to \pi_{k-1}(O_{k+1})))$, $i^k: O_k \to \mathrm{PL}_k$ the natural map.

By standard smoothing theory [17],

(3) $\pi_i(\mathrm{PL}_k/O_k) \to \pi_i(\mathrm{PL}/O)$ is bijective for $i \leq k$.

By Theorem 4.6 (see also Hirsch [38]),

(4) $\pi_{n+1}(\mathrm{PL}_n/O_n) \to \pi_{n+1}(\mathrm{PL}/O)$ is bijective, $n \geq 5$.

(4′) $\pi_{n+2}(\mathrm{PL}_n/O_n) \to \pi_{n+2}(\mathrm{PL}/O)$ is surjective, $n \geq 5$.

Combining these results we get

THEOREM 5.1. (a) *$\pi_i(\mathrm{PL}_{k+1}/O_{k+1}, \mathrm{PL}_k/O_k) = 0$ for $i \leq k+2$, $k \geq 5$.*

(b) *$\pi_i(\mathrm{Top}_{k+1}/O_{k+1}, \mathrm{Top}_k/O_k) = 0$ for $i \leq k+2$, $k \geq 5$.*

Part (b) follows from (a) and the results of Kirby and Siebenmann [15].

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Recently, Volodin [35] has defined Whitehead groups $\mathrm{Wh}^0(G)$, $\mathrm{Wh}^1(G)$, $\mathrm{Wh}^2(G), \cdots$ for any group G, with $\mathrm{Wh}^0(G) = \tilde{K}_0(G)$, $\mathrm{Wh}^1(G) = \mathrm{Wh}(G)$ and $\mathrm{Wh}^2(G)$ being the same as the algebraic Whitehead group introduced by Hatcher and Wagoner [12]. Volodin obtains the results of Hatcher and Wagoner on $\pi_0(C^d(M, \partial))$ and also claims $\pi_1(C(M, \partial)) \simeq Z_2 \oplus \mathrm{Wh}^3(0)$, where $C^d(M, \partial)$ = group of concordances (i.e., pseudo-isotopies) of M mod ∂; for M 2-connected and dim M sufficiently large. This last result has also been obtained by Hatcher. In particular, Volodin claims

$$\pi_1(C^d(D^{k-1}, \partial)) = \pi_1(\mathrm{Diff}(D^k, D_+^{k-1})) \simeq Z_2 \oplus \mathrm{Wh}^3(0)$$

for $k \geq 8$. From Theorem 4.4(c) we therefore have

Theorem 5.1. (a$'$) $\pi_{k+3}(\mathrm{PL}_{k+1}/O_{k+1}, \mathrm{PL}_k/O_k) \simeq Z_2 \oplus \mathrm{Wh}^3(0)$, $k \geq 7$.

(b$'$) $\pi_{k+3}(\mathrm{Top}_{k+1}/O_{k+1}, \mathrm{Top}_k/O_k) \simeq Z_2 \oplus \mathrm{Wh}^3(0)$, $k \geq 7$.

Corollary 5.2. (a) $\pi_i(\mathrm{PL}/O, \mathrm{PL}_n/O_n) = 0$ *for* $i \leq n+2$, $n \geq 5$ *and* $\pi_{n+3}(\mathrm{PL}/O, \mathrm{PL}_n/O_n) = Z_2 \oplus \mathrm{Wh}^3(0)$ *if* $n \geq 7$.

(b) $\pi_i(\mathrm{Top}/O, \mathrm{Top}_n/O_n) = 0$ *for* $i \leq n+2$, $n \geq 5$ *and* $\pi_{n+3}(\mathrm{Top}/O, \mathrm{Top}_n/O_n) = Z_2 \oplus \mathrm{Wh}^3(0)$ *if* $n \geq 7$.

Since $\pi_i(\mathrm{PL}_{k+1}/O_{k+1}, \mathrm{PL}_k/O_k) \simeq \pi_i(\mathrm{PL}_{k+1}/\mathrm{PL}_k, O_{k+1}/O_k)$, we have

Theorem 5.1$'$. $\pi_i(\mathrm{PL}_{k+1}/\mathrm{PL}_k, O_{k+1}/O_k) \simeq 0$ *for* $i \leq k+2$, $k \geq 5$, *and* $\pi_{k+3}(\mathrm{PL}_{k+1}/\mathrm{PL}_k, O_{k+1}/O_k) \simeq Z_2 \oplus \mathrm{Wh}^3(0)$, $k \geq 7$. (*Similarly for* Top.)

The main result of [39] gives a fibration $\mathrm{PL}_k \subset \mathrm{PL}_{k+1} \to S^k \times BC^{pl}(S^k)$, where $C^{pl}(S^k)$ is the group of PL pseudo-isotopies of S^k and $BC^{pl}(S^k)$ is the universal base space. Since the fibration $O_k \to O_{k+1} \to S^k$ maps into the above fibration, we have that

Proposition 5.3. $\pi_i(O_{k+1}/O_k) \to \pi_i(\mathrm{PL}_{k+1}/\mathrm{PL}_k)$ *is split injective, all* i, k (*i.e.*, $\mathrm{PL}_{k+1}/\mathrm{PL}_k \simeq O_{k+1}/O_k \times BC^{pl}(S^k)$).

Before proceeding further, we quote other known results:

(5) $\pi_k(O_n) \to \pi_k(O)$ is a split surjection for $k < 2(n-1)$, $n \geq 13$ [30].

(6) $\pi_i(O_n, O_{n-k}) \to \pi_i(G_n, G_{n-k})$ is an isomorphism for all $i < 2(n-k)-2$, where G_n is the H-space of homotopy equivalences of S^{n-1} with the C-O topology [36].

(7) $\pi_i(O) \to \pi_i(\mathrm{PL})$ and $\pi_i(O) \to \pi_i(\mathrm{Top})$ are injective. Also $\pi_i(\mathrm{PL}/O)$ and $\pi_i(\mathrm{Top}/O)$ are finite [40].

(8) $\pi_i(O, O_n) \otimes Q \to \pi_i(G, G_n) \otimes Q$ is an isomorphism for $i \leq 2n-2$ [36].

Proposition 5.4. (i) $\pi_i(O, O_n) \to \pi_i(\mathrm{PL}, \mathrm{PL}_n)$ *and* $\pi_i(O, O_n) \to \pi_i(\mathrm{Top}, \mathrm{Top}_n)$ *are injective for all* $i \leq 2n-2$.

(ii) $\pi_i(O, O_n) \to \pi_i(\mathrm{PL}, \mathrm{PL}_n) \simeq \pi_i(\mathrm{Top}, \mathrm{Top}_n)$ *is bijective for* $i \leq n+2$ *and* $n \geq 5$.

(iii) $\pi_i(O_n) \otimes Q \to \pi_i(\mathrm{PL}_n) \otimes Q$ *and* $\pi_i(O_n) \otimes Q \to \pi_i(\mathrm{Top}_n) \otimes Q$ *are injective for all* i.

(iv) *If* $n \geq 5$, $\pi_i(O_n) \to \pi_i(\mathrm{PL}_n)$ *and* $\pi_i(O_n) \to \pi_i(\mathrm{Top}_n)$ *are injective for* $i \leq n$ *and also for* $i = n+3$ *if* $n \equiv 1, 5, 7 \pmod 8$ *and* $i = n+4$ *if* $n \equiv 0, 4, 5 \pmod 8$.

Proof. (i) follows from (6), and (ii) follows from Corollary 5.2. For (iii) consider the commutative diagram

$$(9)\qquad \begin{tikzcd}
 & {} \arrow[r] & \pi_{i+1}(O/O_n) \arrow[r, "\partial"] \arrow[d] & \pi_i(O_n) \arrow[r, "j"] \arrow[d] & \pi_i(O) \arrow[r] \arrow[d] & {} \\
 & {} \arrow[r] & \pi_{i+1}(\mathrm{PL}/\mathrm{PL}_n) \arrow[r, "\partial"] \arrow[d] & \pi_i(\mathrm{PL}_n) \arrow[r, "j"] \arrow[d] & \pi_i(\mathrm{PL}) \arrow[r] \arrow[d] & {} \\
 & \pi_{i+1}(G) \arrow[r] & \pi_{i+1}(G/G_n) \arrow[r, "\partial"] & \pi_i(G_n) \arrow[r, "j"] & \pi_i(G) \arrow[r] & {}
\end{tikzcd}$$

Since $\pi_i(G) \otimes Q = 0$ all i, and $\pi_i(O) \otimes Q \to \pi_i(\mathrm{PL}) \otimes Q$ is an isomorphism by (7), it follows from (8) that $\pi_i(O_n) \otimes Q \to \pi_i(\mathrm{PL}_n) \otimes Q$ is injective for $i \leq 2n-3$. But $\pi_i(O_n) \otimes Q = 0$ for $i > 2n-3$.

For the first part of (iv) note first that for $i \leq n-2$, $\pi_i(O_n) \to \pi_i(O)$ is an isomorphism and hence by (7), $\pi_i(O_n) \to \pi_i(\mathrm{PL}_n)$ is a monomorphism. Now consider the commutative diagram:

$$\begin{tikzcd}
{} \arrow[r] & \pi_i(O_{n+1}/O_n) \arrow[r, "\partial_*"] \arrow[d] & \pi_i(O_n) \arrow[r, "j"] \arrow[d] & \pi_i(O_{n+1}) \arrow[d] \\
{} \arrow[r] & \pi_i(\mathrm{PL}_{n+1}/\mathrm{PL}_n) \arrow[r, "\partial_*"] \arrow[d] & \pi_i(\mathrm{PL}_n) \arrow[r, "j"] \arrow[d] & \pi_i(\mathrm{PL}_{n+1}) \arrow[d] \\
{} \arrow[r] & \pi_i(G_{n+1}/F_{n+1}) \arrow[r, "\partial_*"] & \pi_i(F_{n+1}) \arrow[r, "j"] & \pi_i(G_{n+1}) \arrow[r] & {}
\end{tikzcd}$$

If $i = n-1$, and $\alpha \in \pi_{n-1}(O_n)$ is in Ker $j = \operatorname{Im} \partial$, then α is either infinite cyclic or of order 2. The first case is covered by (iii), and in the second case it maps nontrivially into $\pi_{n-1}(F_{n+1})$, since for i_n the generator of $\pi_n(S^n) = \pi_n(G_{n+1}/F_{n+1})$, $\partial i_n \in \pi_{n-1}(F_{n+1})$ is nontrivial (except for $n = 1, 3, 7$). Hence in either case Ker $j \subset \pi_{n-1}(O_n)$ maps monomorphically into $\pi_{n-1}(\mathrm{PL}_n)$; and if $\alpha \in \pi_{n-1}(O_n)$ maps nontrivially under j, it maps nontrivially into $\pi_{n-1}(\mathrm{PL}_n)$ by the first part of the argument.

For $i = n$ and $n \equiv 3 \pmod 4$ the result follows from the fact that $\pi_{n+1}(O/O_n) = 0$ [33].

If $n \not\equiv 3 \pmod 4$, let $\eta_n \in \pi_{n+1}(S^n) = Z_2$, the generator. Then under the map $\pi_{n+1}(S^n) \xrightarrow{\partial} \pi_n(F_{n+1}) \simeq \pi_{2n}(S^n)$, η_n goes into the Whitehead product $[\eta_n, i_n] \neq 0$ [32]. Combining this with the injectivity of $\pi_n(O_{n+1}) \to \pi_n(\mathrm{PL}_{n+1})$, the result follows.

Finally, the second part of (iv) follows because for $n \geq 4$, $\pi_{i+n}(O, O_n) = 0$ for $i = 4$ and $n \equiv 1, 5, 7 \pmod 8$, or for $i = 5$ and $n \equiv 0, 4, 5 \pmod 8$, see [33].

The argument for Top_n is identical.

Corollary 5.5. *If $n \geq 5$ and $i \leq n + 1$, $\pi_i(\mathrm{PL}_n) \to \pi_i(\mathrm{PL})$ and $\pi_i(\mathrm{Top}_n) \to \pi_i(\mathrm{Top})$ are surjective.*

Proof. It suffices to show that $\pi_{i+1}(\mathrm{PL}/\mathrm{PL}_n) \to \pi_i(\mathrm{PL}_n)$ is injective for $i \leq n$. But this follows from Proposition 5.4(ii) and (iv), using diagram (9).

Now we compare PL_n, Top_n with the corresponding groups $\widetilde{\mathrm{PL}}_n$, $\widetilde{\mathrm{Top}}_n$ for block bundles (see Part I). In the limit, $\mathrm{PL} = \widetilde{\mathrm{PL}}$, $\mathrm{Top} = \widetilde{\mathrm{Top}}$. Also by [26], $\pi_i(O/O_n) \simeq \pi_i(\mathrm{PL}/\widetilde{\mathrm{PL}}_n) \simeq \pi_i(\mathrm{Top}/\widetilde{\mathrm{Top}}_n) \simeq \pi_i(G/G_n)$ for $n \geq 5$, $i < 2n - 2$.

Proposition 5.6. *If $n \geq 5$, $\pi_i(\mathrm{PL}_n) \to \pi_i(\widetilde{\mathrm{PL}}_n)$ and $\pi_i(\mathrm{Top}_n) \to \pi_i(\widetilde{\mathrm{Top}}_n)$ are isomorphisms for $i \leq n + 1$ and epimorphisms for $i = n + 2$.*

Proof. Consider the commutative diagram

$$\begin{array}{ccccccccc}
\to \pi_{i+1}(\mathrm{PL}/O, \mathrm{PL}_n/O_n) & \to & \pi_i(\mathrm{PL}_n/O_n) & \to & \pi_i(\mathrm{PL}/O) & \to & \pi_i(\mathrm{PL}/O, \mathrm{PL}_n/O_n) \to \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
\to \pi_{i+1}(\mathrm{PL}/O, \widetilde{\mathrm{PL}}_n/O_n) & \to & \pi_i(\widetilde{\mathrm{PL}}_n/O_n) & \to & \pi_i(\mathrm{PL}/O) & \to & \pi_i(\mathrm{PL}/O, \widetilde{\mathrm{PL}}_n/O_n) \to
\end{array}$$

We have

$$\pi_i(\mathrm{PL}/O, \widetilde{\mathrm{PL}}_n/O_n) \cong \pi_i(\mathrm{PL}/\widetilde{\mathrm{PL}}_n, O/O_n) = 0 \quad \text{for } i < 2n - 2,$$

$$\pi_i(\mathrm{PL}/O, \mathrm{PL}_n/O_n) \cong \pi_i(\mathrm{PL}/\mathrm{PL}_n, O/O_n) = 0 \quad \text{for } i \leq n + 2$$

by 5.2. Hence $\pi_i(\mathrm{PL}_n/O_n) \to \pi_i(\widetilde{\mathrm{PL}}_n/O_n)$ is an isomorphism for $i \leq n + 1$ and an epimorphism for $i = n + 2$. The result follows.

Corollary 5.7. *If $n \geq 5$ and $M^n \subset M^{n+k}$ is a PL embedding of PL manifolds, then M^n has a PL normal bundle if $n \leq k + 3$, which is unique if $n \leq k + 2$.*

Conjecture.(3) $\pi_i(O_n) \to \pi_i(\mathrm{PL}_n)$ is injective for all $i < 2n - 3$.

We note that the kernel is at most 2-primary. This is well known and we give a simple argument: In fact, $\pi_i(G_n) \to \pi_i(G)$ is an epimorphism for $i \leq 2n - 3$, mod

(3) A. Campo informs us that he can prove $\pi_i(O_n) \to \pi_i(\mathrm{PL}_n)$ is injective for $i \leq n+2$.

the 2-primary component. Hence from the diagram

$$\begin{tikzcd}
\pi_i(\widetilde{\mathrm{PL}}_n) \arrow[r] \arrow[d] & \pi_i(G_n) \arrow[d] \\
\pi_i(\mathrm{PL}) \arrow[r] \arrow[d] & \pi_i(G) \arrow[d] \\
\pi_i(\mathrm{PL}/\widetilde{\mathrm{PL}}_n) \arrow[r, phantom, "\simeq"] & \pi_i(G/G_n)
\end{tikzcd}$$

we see that $\pi_i(\widetilde{\mathrm{PL}}_n) \to \pi_i(\mathrm{PL})$ is an epimorphism mod the 2-primary component. But then from the diagram

$$\begin{tikzcd}
 & \pi_{i+1}(O/O_n) \arrow[r, phantom, "\simeq"] \arrow[d] & \pi_{i+1}(\mathrm{PL}/\widetilde{\mathrm{PL}}_n) \arrow[d] \\
 & \pi_i(O_n) \arrow[r] \arrow[d] & \pi_i(\widetilde{\mathrm{PL}}_n) \arrow[d] \\
0 \arrow[r] & \pi_i(O) \arrow[r] & \pi_i(\mathrm{PL})
\end{tikzcd}$$

we see that $\pi_i(O_n) \to \pi_i(\widetilde{\mathrm{PL}}_n)$ is injective mod 2-primary for $i < 2n = 3$, and *a fortiori*, the same holds for $\pi_i(O_n) \to \pi_i(\mathrm{PL}_n)$.

6. **Isotopy and pseudo-isotopy.** If M^n is a compact differentiable manifold and $t\colon |M^n| \to M^n$ a differentiable triangulation; let $C^d(M, \partial)$, $C^{pl}(M, \partial)$, $C^t(M, \partial)$ denote the groups

$$\mathrm{Diff}(M \times I;\, \partial M \times I \cup M \times \{0\}), \qquad \mathrm{PL}(|M| \times J,\, \partial|M| \times I \cup |M| \times \{0\}),$$

$$\mathrm{Homeo}(M \times I;\, \partial M \times I \cup M \times \{0\})$$

respectively. (In the differentiable case one needs to "smooth the corners" of $M \times I$ if $\partial M \neq \emptyset$.) These groups are the so-called *concordance* or *pseudo-isotopy* groups.

Theorem 6.1. *If M^n is a compact connected differentiable manifold, then $\pi_i(C^d(M, \partial)) \to \pi_i(C^{pl}(M, \partial))$ is an isomorphism for $i = 0$ and an epimorphism for $i = 1$.*

In particular, this theorem says that a differentiable pseudo-isotopy of $(M, \partial M)$ may be deformed into an isotopy if and only if it may be so deformed as a piecewise differentiable pseudo-isotopy.

Theorem 6.2. *If M^n is a compact connected piecewise linear manifold with $n \geq 5$, then $\pi_i(C^{pl}(M, \partial)) \to \pi_i(C^t(M, \partial))$ is an isomorphism, all i.*

To prove these theorems, we will apply the main theorem (Theorem 4.2)(4) of Part I, to show that $\pi_i(C^{pl}(M, \partial))/C^d(M, \partial) = 0$ for $i = 0, 1$ and $\pi_i(C^t(M, \partial))/(C^{pl}(M, \partial)) = 0$, all i. For this purpose we need some elementary lemmas on the space of sections of a fibre bundle which we state without proof.

Lemma 6.3. *Let $p: P \to B$ be a fibre bundle with connected fibre F. Let $\Gamma(P)$ = space of sections of P. Suppose given a fixed section $s_0: B \to P$. Then*

(a) *If p is trivial, $\Gamma(P) = F^B$ and we have a split fibration*

$$(F, *)^{(B,*)} \to F^B \rightleftarrows F.$$

(b) *If B is a suspension of a CW-complex, we have a fibration*

$$(F, *)^{(B,*)} \to \Gamma(P) \to F.$$

(c) *If F is k-connected and B is a CW-complex with* $\dim B = j$, *then $\pi_i(\Gamma(P)) = 0$ for $i + j \leq k$.*

Lemma 6.4. *Let $p: P \to B$ be a fibre bundle with connected fibre F, and $p_1: P_1 \to B$ a subbundle $P_1 \subset P$ with connected fibre $F_1 \subset F$. Let $\Gamma^{\partial_0}(P \times I, P_1)$ = space of sections s of $p \times 1: P \times I \to B \times I$ such that $s \mid B \times 0 = s_0$ and $s(B \times 1) \subset P_1 \times 1$. Then*

(a) *If (p, p_1) is trivial, $\Gamma^{\partial_0}(P \times I, P_1) = E(F, F_1)^B$ and we have the split fibration:*

$$(E(F, F_1), *)^{(B,*)} \to (E(F, F_1))^B \rightleftarrows E(F, F_1).$$

(Here $E(F, F_1)$ is the space of paths in E from base point to F_1.)

(b) *If B is a suspension of a CW-complex, we get a fibration*

$$(E(F, F_1), *)^{(B,*)} \to \Gamma^{\partial_0}(P \times I, P_1) \to E(F, F_1).$$

In particular, if $P_1 = s_0(B)$, then we get a fibration

$$(\Omega(F), *)^{(B,*)} \to \Gamma^{\partial_0}(P \times I, P_1) \to \Omega F.$$

(c) *If $\pi_i(E(F, F_1)) \simeq \pi_{i+1}(F, F_1) = 0$ for $i \leq k$, and* $\dim B = j$, *then $\pi_i(\Gamma^{\partial_0}(P, P_1)) = 0$ for $i + j \leq k$.*

Proof of 6.1. By Theorem 4.2 of Part I,

$$C^{pl}(M, \partial)/C^d(M, \partial) = \Gamma^{\partial_0}(P^{pl} \times I, \partial P^{pl}, s_0),$$

where the fibre of P^{pl} is $\mathrm{PL}_{n+1}/O_{n+1}$ and the fibre of ∂P^{pl} is PL_n/O_n. By Theorem 5.1$'$ we have

(4) We use the Top-PL version of Theorem 4.2 to prove Theorem 6.2.

$$\pi_i(E(\mathrm{PL}_{n+1}/O_{n+1}, \mathrm{PL}_n/O_n)) = 0 \quad \text{for } i \leq n+1.$$

Hence by Lemma 6.4(c), $\pi_i(C^{pl}(M, \partial)/C^d(M, \partial)) = 0$ for $i = 0, 1$.

Proof of 6.2. The argument is the same, except in this case

$$\pi_i(\mathrm{Top}_{n+1}/\mathrm{PL}_{n+1}, \mathrm{Top}_n/\mathrm{PL}_n) = 0 \quad \text{all } i \text{ if } n \geq 5.$$

We now consider the special case of $C^d(S^n)$ and $C^{pl}(S^n)$.

By Proposition 5.3,

(1) $$C^{pl}(S^n) \simeq E(\mathrm{PL}_{n+1}/O_{n+1}, \mathrm{PL}_n/O_n).$$

Hence by Theorem 5.1′,

(2) $$\begin{aligned} \pi_i(C^{pl}(S^n)) &= 0 \quad \text{for } i \leq n+1,\ n \geq 5 \\ &= Z_2 + \mathrm{Wh}^3(0) \quad \text{for } i = n+2,\ n \geq 7. \end{aligned}$$

On the other hand, Chenciner [37] has shown that $\mathrm{Diff}(D^{n+1}, D^n_+) \simeq C^d(S^n)$ and hence by Theorem 4.4

(3) $$C^d(S^n) \simeq \Omega^{n+1}(E(\mathrm{PL}_{n+1}/O_{n+1}, \mathrm{PL}_n/O_n)),$$

and hence by (1) and (3) with $C^d_n = C^d(S^n)$, $C^{pl}_n = C^{pl}(S^n)$.

Proposition 6.5 (Chenciner). *$\pi_i(C^d_n) \simeq \pi_{i+n+1}(C^{pl}_n)$, all i.*

Thus we see that, although $\pi_0(C^{pl}(M, \partial)) \simeq \pi_0(C^d(M, \partial))$, $\pi_1(C^{pl}(M, \partial)) \not\simeq \pi_1(C^d(M, \partial))$ in general. The relationship between PL and Diff pseudo-isotopy groups will be explored in a forthcoming paper with M. Rothenberg.

Finally, we consider the implications of these results for diffeomorphisms of the disc: First Volodin and Hatcher have pointed out that the split fibration $C^d(S^{n-1}) \to \mathrm{Diff}\, D^n \rightleftarrows SO_n$, together with their result on $\pi_1(C^d(S^{n-1}))$ implies

(4) $$\pi_0(\mathrm{Diff}\, D^n) = 0 \quad \text{and} \quad \pi_1(\mathrm{Diff}\, D^n) \simeq Z_2 + Z_2 + \mathrm{Wh}^3(0).$$

(We are considering only orientation preserving diffeomorphisms.)

To obtain information on $\mathrm{Diff}(D^n, \partial)$, we consider the fibration

(5) $$\mathrm{Diff}(D^{n+1}, \partial) \to \mathrm{Diff}(D^{n+1}, D^n_+) \to \mathrm{Diff}(D^n, \partial)$$

and its homotopy sequence

$$\begin{aligned} &\to \pi_2(\mathrm{Diff}(D^n, \partial)) \to \pi_1(\mathrm{Diff}(D^{n+1}, \partial)) \to \pi_1(\mathrm{Diff}(D^{n+1}, D^n_+)) \to \pi_1(\mathrm{Diff}(D^n, \partial)) \\ &\qquad \to \pi_0(\mathrm{Diff}(D^{n+1}, \partial)) \to \pi_0(\mathrm{Diff}(D^{n+1}, D^n_+)). \end{aligned}$$

Since $\pi_0(\mathrm{Diff}(D^{n+1}, D^n_+)) = 0$ and $\pi_0(\mathrm{Diff}(D^{n+1}, \partial))$ is Milnor's Γ_{n+1} group and $\pi_1(\mathrm{Diff}(D^{n+1}, D^n_+)) \cong Z_2 + \mathrm{Wh}^3(0)$, we see that $\pi_1(\mathrm{Diff}(D^n, \partial))$ is arbitrarily large, and hence is general $\pi_1(\mathrm{Diff}(D^{n+1}, \partial)) \to \pi_1(\mathrm{Diff}(D^{n+1}, D^n_+))$ has a kernel. Hence $\pi_2(\mathrm{Diff}(D^n, \partial)) \neq 0$ in general.

7. A bilinear pairing and some new homotopy of $\mathrm{Diff}(D^n, \partial D^n)$. In this chapter, using the results of §4, we will get some estimate on the homotopy groups of $\mathrm{Diff}(D^n, \partial D^n)$, Theorem 7.4. The methods of [2], [3] permit the use of these estimates to prove the nontriviality of $\pi_i(\mathrm{Diff}(M^n))$ (see for instance Corollary 7.5). From now on we assume that we are in the category of basepointed spaces which have abelian fundamental groups.

Let us consider the map $\psi^X_{k,p}: \pi_k(X, *) \times \pi_p(S^k, s_0) \to \pi_p(X, *)$ defined by $\psi(\alpha, \beta) = \alpha_{\#}(\beta)$; more precisely, if α is a homotopy class $(S^k, s_0) \to (X, *)$ it induces a group homomorphism α for homotopy groups in dimension p, and we take as $\psi(\alpha, \beta)$ the image of β by this homomorphism. With this definition, it is clear that

$$\psi^X_{k,p}(\alpha, \beta_1 + \beta_2) = \psi^X_{k,p}(\alpha, \beta_1) + \psi^X_{k,p}(\alpha, \beta_2).$$

If we set $\dot{\pi}_p(S^k) = \{\beta \in \pi_p(S^k) \mid \beta \in \mathrm{Im}\ \Sigma\}$, where by Σ we mean the suspension homomorphism, and if we consider the restriction of $\psi^X_{k,p}$ to $\pi_k(X, *) \times \dot{\pi}_p(S^k) \to \pi_p(X, *)$ we also have

$$\psi^X_{k,p}(\alpha_1 + \alpha_2, \beta) = \psi^X_{k,p}(\alpha_1, \beta) + \psi^X_{k,p}(\alpha_2, \beta)$$

for any β. Indeed if $\beta \in \dot{\pi}_p(S^k)$, it can be represented by $\Sigma\beta': S^p \to S^k$ with $\beta': (S^{p-1}, s_0) \to (S^{k-1}, s_0)$; such a map induces always a group homomorphism $(\Sigma\beta')_*: \pi_k(X, *) \to \pi_p(X, *)$, and because $\psi^X_{k,p}(\alpha, \beta) = (\Sigma\beta')_*(\alpha)$ we have

$$\psi^X_{k,p}(\alpha_1 + \alpha_2, \beta) = \psi^X_{k,p}(\alpha_1, \beta) + \psi^X_{k,p}(\alpha_2, \beta).$$

Therefore we can extend $\psi^X_{k,p}$ to a unique group homomorphism $\bar{\psi}^X_{k,p}: \pi_k(X, *) \otimes \dot{\pi}_p(S^k, s_0) \to \pi_{k+p}(X, *)$. It is easy to check that $\bar{\psi}^X \cdots$ is natural with respect to X, i.e., if $f: (X, *) \to (Y, *)$ is a continuous map, we have the following commutative diagram

$$\begin{array}{ccc}
\pi_k(X, *) \otimes \dot{\pi}_p(S^k, s_0) & \xrightarrow{\bar{\psi}^X_{k,p}} & \pi_p(X, *) \\
{\scriptstyle f_*}\downarrow \quad\quad \downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle f_*} \\
\pi_k(Y, *) \otimes \dot{\pi}_p(S^k, s_0) & \xrightarrow{\bar{\psi}^Y_{k,p}} & \pi_p(Y, *)
\end{array}$$

We also notice that for $p \leq 2k - 2$, $\pi_p(S^k, s_0) = \pi_{p+1}(S^{k+1}, s_0) = \pi^S_{p-k}$, the

kth stable homotopy group of spheres. This means that, for any (k, l), $l \leq k-2$, we have a group homomorphism $\phi^X_{k,l}: \pi_k(X, *) \otimes \pi^S_l \to \pi_{k+l}(X, *)$ defined by $\phi^X_{k,l} = \bar{\psi}^X_{k,l+k}$ which is natural with respect to X. More generally, given $(k, l_1, \cdots, l_r)$ with $l_1 \leq k-2$, $l_2 \leq k + l_1 - 2, \cdots, l_r \leq k + l_1 + \cdots + l_{r-1} - 2$, we can define inductively the group homomorphism

$$\phi^X_{k;l_1,l_2,\cdots,l_r}: \pi_k(X, *) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \to \pi_{k+l_1+\cdots+l_r}(X, *)$$

by

$$\phi^X_{k;l_1,\cdots,l_r} = \phi^X_{k+l_1+\cdots+l_{r-1},l_r}(\phi^X_{k;l_1,\cdots,l_{r-1}} \otimes \mathrm{id}).$$

$\phi^X_{k,l_1,\cdots,l_r}$ is clearly natural with respect to X. If we apply this homomorphism to $X = \mathrm{PL}_n/O_n$ and PL/O, and one uses the homotopy equivalence $\mathrm{Diff}(D^n, \partial D^n) \simeq \Omega^{n+1}(\mathrm{PL}_n/O_n)$, we obtain

Theorem 7.1. *For any n and any sequence of natural numbers $k, l_1, \cdots, l_r$ with the property* (*), $l_1 \leq k-2$, $l_2 \leq k + l_1 - 2, \cdots, l_r \leq k + l_1 + \cdots + l_{r-1} - 2$ *there exists a group homomorphism*

$$\psi_{k;l_1,\cdots,l_r}: \pi_{k-n-1}(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \to \pi_{k-n-1+l_1+\cdots+l_r}(\mathrm{Diff}(D^n, \partial D^n)),$$

so that the following diagram is commutative:

$$\begin{tikzcd}
\pi_{k-n-1}(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r, "\psi^n_{k,l_1,\cdots,l_r}"] \arrow[d, "\gamma^n_k" left] & \pi_{k-n-1+l_1+\cdots+l_r}(\mathrm{Diff}(D^n, \partial D^n)) \arrow[d, "\gamma^n_{k+l_1+\cdots+l_r}"] \\
\pi_k(\mathrm{PL}/O) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r, "\phi^{\mathrm{PL}/O}_{k,l_1,\cdots,l_r}"] & \pi_{k+l_1+\cdots+l_r}(\mathrm{PL}/O)
\end{tikzcd}$$

where $\gamma^n_p: \pi_{p-n-1}(\mathrm{Diff}(D^n, \partial D^n)) \to \pi_p(\mathrm{PL}/O)$ *is the homomorphism induced for homotopy groups by* $\mathrm{Diff}(D^n, \partial D^n) \simeq \Omega^{n+1}(\mathrm{PL}_n/O_n) \to \Omega^{n+1}(\mathrm{PL}/O)$.

Proof. We take $\psi^n_{k,l_1,\cdots,l_r} = \phi^{PL_n/O_n}_{k,l_1,\cdots,l_r}$ via the isomorphism

$$\pi_{s-n-1}(\mathrm{Diff}(D^n, \partial D^n)) \cong \pi_s(\mathrm{PL}_n/O_n).$$

Remark. For $r = 1$ and $J_l: \pi_l(SO) \to \pi^S_l$ the J-homomorphism

$\psi^n_{k,l}(\mathrm{id} \otimes J_l)\colon \pi_{k-n-1}(\mathrm{Diff}(D^n, \partial D^n)) \otimes \pi_l(SO) \to \pi_{k-n-1+l}(\mathrm{Diff}(D^n, \partial D^n))$ $(l \leq k-2)$ seems to be the so-called Munkres-Milnor-Novicov pairing see [2]. It seems also that $\phi^{\mathrm{PL}/O}_{k,l_1}$ is the so-called Bredon pairing (see [27]). As we are not interested, we do not prove the equivalence between this construction and the mentioned pairings which are originally defined in very geometric terms. In what follows we will use Theorem 7.1 to get estimates about $\pi_*(\mathrm{Diff}(D^n, \partial D^n))$. We should mention that John Grier Miller [28] has used the Bredon pairing (in the original Bredon definition) to detect nontrivial elements in $\pi_*(\mathrm{Diff}(D^n, \partial D^n))$. His geometric point of view is different from the present, and the range of (k, n) in $\pi_k(\mathrm{Diff}(D^n, \partial D^n))$ for which he can detect nontrivial elements is essentially less than ours.

In the computations below, we will start with $k = n + 2$ because for $n \geq 5$ by Theorem 4.6(Ap), γ^n_{n+2} is an epimorphism so it suffices to check $\phi^{\mathrm{PL}/O}_{n+2,l_1+\cdots+l_r}$ is nontrivial in order to conclude that $\pi_{1+l_1+\cdots+l_j}(\mathrm{Diff}(D^n, \partial D^n))$ is nontrivial (in fact, in order to conclude $\pi_{1+l_1+\cdots+l_j-s}(\mathrm{Diff}(D^{n+s}, \partial D^{n+s}))$ is nontrivial). For $k = n + 2$ the condition (*) becomes

$$(**) \quad l_1 \leq n, \quad l_2 \leq n + l_1, \quad l_3 \leq n + l_1 + l_2, \ \cdots, \ l_r \leq n + l_1 + \cdots + l_{r-1}.$$

Let us consider the following commutative diagram:

$$\begin{tikzcd}
\pi_k(\mathrm{PL}_n/O_n) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r, "\phi^{\mathrm{PL}_n/O_n}"] \arrow[d] \arrow[dd, bend right] & \pi_{k+l_1+\cdots+l_r}(\mathrm{PL}_n/O_n) \arrow[d] \\
\pi_k(\mathrm{PL}/O) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r] \arrow[d, "q_k", "\mathrm{id}"'] & \pi_{k+l_1+\cdots+l_r}(\mathrm{PL}/O) \arrow[d, "q_{k+l_1+\cdots+l_r}"] \\
\pi_k(G/O) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r] & \pi_{k+l_1+\cdots+l_r}(G/O) \\
\pi_k(G) \otimes \pi^S_{l_1} \otimes \cdots \otimes \pi^S_{l_r} \arrow[r, "\phi^G_{k,l_1,\cdots,l_r}"] \arrow[u, "P_k"] & \pi_{k+l_1+\cdots+l_r}(G) \arrow[u, "P_{k+l_1+\cdots+l_r}"']
\end{tikzcd}$$

where G denotes the topological semigroup $G = \operatorname{inj\,lim} G_n$, where G_n is the space of all continuous homotopy equivalences of S^{n-1}. It is well known that $\pi_i(G) = \pi^S_i$ and easy to check that $\phi^G_{k,l_1,\cdots,l_r}$ viewed as the group homomorphism

$\phi^G_{k,l_1,\cdots,l_r}: \pi^S_k \otimes \cdots \otimes \pi^S_{l_r} \to \pi^S_{k+l_1+\cdots+l_r}$ is the standard composition (multiplication) in π^S_*.

We also notice that P_k is surjective for all $k \not\equiv 0 \pmod 4$ and q_k is surjective[5] (mod 2) for all $k \not\equiv 0 \pmod 4$ and surjective for all $k \not\equiv 2^l - 2$ [29].

Proposition 7.2. *If $n \geq 5$, $n \not\equiv 2 \pmod 4$, $l_1, \cdots, l_r$ is a sequence of natural numbers so that $l_1 \leq n$, $l_2 \leq n + l_1, \cdots, l_r \leq n + l_1 + \cdots + l_{r-1}$ and $(P_{n+2+l_1+\cdots+l_r} \cdot \phi^G_{n+2,l_1,\cdots,l_r}) \not\equiv 0 \pmod 2$ or $n \not\equiv 2^i - 4$ and $P_{n+2+l_1+\cdots+l_r} \cdot \phi^G_{n+2,l_1,\cdots,l_r} \neq 0$, then*

$$\pi_{1+l_1+\cdots+l_r-s}(\mathrm{Diff}(D^{n+s}, \partial D^{n+s})) \neq 0.$$

Moreover, $\gamma^{n+s}_{n+2+l_1+\cdots+l_r} \neq 0$ for any $s < 1 + l_1 + \cdots + l_r$.

It is clear that it suffices to prove Proposition 7.2 only for $s = 0$ because $\gamma^n_{n+2+l_1+\cdots+l_r} \neq 0$ implies

$$\pi_{1+l_1+\cdots+l_r}(\mathrm{PL}_n/O_n) \to \pi_{1+l_1+\cdots+l_r}(\mathrm{PL}/O)$$

is different from zero and therefore

$$\pi_{1+l_1+\cdots+l_r}(\mathrm{PL}_{n+s}/O_{n+s}) \to \pi_{1+l_1+\cdots+l_r}(\mathrm{PL}/O)$$

is different from zero, which is equivalent to saying $\gamma^{n+s}_{n+2+l_1+\cdots+l_r} \neq 0$. The proof for $s = 0$ is immediate from the hypotheses.

To make Proposition 7.2 effective, we recall the following theorem of Toda [24].

Theorem 7.3.(a) *For any prime number p there exist $\alpha_1 \in (\pi^s_{2p-3}; p)$ and $\beta_t \in (\pi^s_{2(tp+t-1)(p-1)-2}; p)$, $1 \leq t \leq p$, so that*

$$(\pi^s_{2(rp+s+1)(p-1)-2(r-s)-1}; p) = Z_p = \{\alpha_1 \cdot \beta_1^{r-s-1} \cdot \beta_{s+1}\}$$

for $0 \leq s < r \leq p-1$ and $r - s \not\equiv p - 1$, where (G, p) denotes the p-primary component of the abelian group G and $\{x\}$ means that the group $\{x\}$ is generated by x.

(b) *If $p \neq 2$ then $P_{2(rp+s+1)(p-1)-2(r-s)-1}(\alpha_1 \cdot \beta_1^{r-s-1} \cdot \beta_{s+1}) \neq 0$.*

We take now $p \neq 2$, $s = 0$, $n = 2p(p-1) - 4$, $l_1 = 2p - 3$, $l_2 = l_3 = \cdots = l_r = 2p(p-1) - 2$, $\beta_1 \in \pi^s_{2p(p-1)-2} = \pi^s_{n+2}$, $\alpha_1 \in \pi^s_{2p-3}$, and then we know

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(5) If $h: G_1 \to G_2$ is a group homomorphism, G_1, G_2 abelian groups, we say "h is surjective mod 2" if the cokernel of h belongs to the class of 2-primary groups and we say h is "zero mod 2" if its image belongs to the class of 2-primary groups.

$$P_{n+2+l_1+\cdots+l_r} \cdot \phi^G_{n+2,l_1,\ldots,l_v}(\beta_1 \cdot \alpha_1 \cdot \beta_1 \cdot \cdots \cdot \beta_1) \neq 0 \pmod 2.$$

Theorem 7.4. *If p is an odd prime, there exists a subgroup*

$$\mathbf{Z}_p \subseteq \pi_{2p+2(r-1)(p^2-p-1)-2-s}(\mathrm{Diff}\, D^{2(p^2-p-2)+s}, \partial D^{2(p^2-p-2)+s})$$

such that

$$\gamma\vdots\vdots/\mathbf{Z}_p\colon \mathbf{Z}_p \to \pi_{2p+2r(p^2-p-1)-3}(\mathrm{PL}/O)$$
$$\to \Theta_{2p+2r(p^2-p-1)-3}/bP_{2p+2r(p^2-p-1)-2}$$

is nontrivial, therefore injective, for any $1 \le r \le p-2$ and any $s < 2p + 2(r-1)(p^2-p-1)-2$.

Proof. It suffices to prove Theorem 7.4 for $s = 0$ because this means

$$\mathbf{Z}_p \subseteq \pi_{2p+2r(p^2-p-1)-3}(\mathrm{PL}_{2(p^2-p-2)}/O_{2(p^2-p-2)})$$
$$\to \pi_{2p+2r(p^2-p-1)-3}(\mathrm{PL}/O) \to \Theta_{2p+2r(p^2-p-1)-3}/bP_{2p+2r(p^2-p-1)-2}$$

is injective, and since the first arrow factors through

$$\pi_{2p+2r(p^2-p-1)-3}(\mathrm{PL}_{2(p^2-p-2)+s}/O_{2(p^2-p-2)+s})$$

we get Theorem 7.4 in its full generality. The proof for $s = 0$ follows immediately from Proposition 7.2 and Theorem 7.3 as soon as we notice that the element $\phi^{\mathrm{PL}_n/O_n}_{\ldots}(\tilde{\beta}_1, \alpha_1, \beta_1, \cdots, \beta_1)$ has to be of order p, because β_1 is. $\tilde{\beta}_1$ is an element in $\pi_{n+2}(\mathrm{PL}_n/O_n)$ which by $q_{n+2}\gamma^n_{n+2}$ goes onto $P_k(\beta_1)$.

Theorem 7.3 is of particular interest for $r = p-2$; in which case we know that

$$\pi_{2(p^3-4p^2+3p+2)-s}\,\mathrm{Diff}(D^{2(p^2-p-2)+s}, \partial D^{\cdots})$$

contains elements of order p which survives in $\Theta_{2p(p^2-3p+2)-1}$. These elements give the first known examples of nontrivial elements in $\pi_k(\mathrm{Diff}\, D^n, \partial D^n)$ which survives in Θ_{n+k+1} for $k > n$.

Remarks. (a) Of course other computations in stable homotopy theory which are compatible with the conditions required by Proposition 7.2 can be successfully used.

(b) The method described cannot be used for detecting elements in $\pi_k(\mathrm{Diff}(D^n, \partial D^n))$ which go by γ^n_k in bP_{n+k+2} (because of the homotopy structure of G/PL).

We can use the result stated by Theorem 7.4 to get nontrivial elements in $\pi_i(\mathrm{Diff}(M^n))$ for quite general manifolds. For instance, combining Theorem 7.4 with the diagram of [3, p. 128] with 3.5.13 we obtain the following.

Theorem 7.5. *Let M^n be a compact differentiable manifold which is a homotopy torus. Then for any $p \neq 2$ and $r < p - 1$ so that $n = 2(p^2 - p - 2) + s$ with $s < 2p + 2(r-1)(p^2 - p - 1) - 2$,*

$$\pi_{2p+2(r-1)(p^2-p-1)-2-s}(\mathrm{Diff}(M^n, X))$$

contains elements of order p.

As a general conclusion about Theorem 7.4, we would mention that given n, $\pi_*(\mathrm{Diff}(D^n, \partial D^n))$, the homotopy of the connected component of identity, contains elements of order $p \neq 2$ if $n \geq 2(p^2 - p - 1)$.

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